Springs and Pendula

The Spring
The Torsion Pendulum
The Simple Pendulum
The Physical Pendulum
Example Problems
Damping & Resonance

SHM Reminder

1. There is an equil pt 2. $F \propto -x$ which means $a = -\omega^2 x$

For real SH oscillators,

We want to be able to calculate ω (and f and T) from the physical characteristics of the oscillator

Simple Harmonic Oscillators must have an equilibrium point, and the restoring force must be proportional to the displacement.

$$F = -kx \qquad \frac{N}{m}$$

The motion will obey:

$$x(t) = x_{max} \cos(\omega t + \phi)$$

A quick word on our method:

You may remember:

$$x(t) = x_{max}\cos(\omega t + \phi)$$

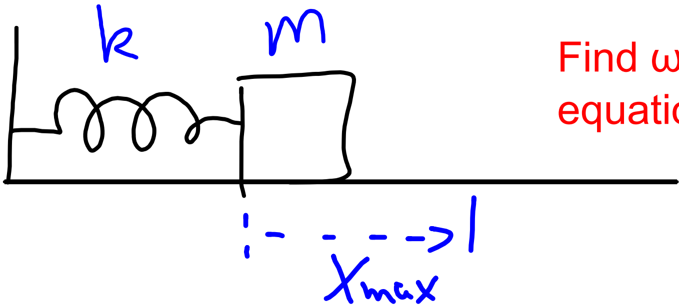
$$v(t) = -\omega x_{max}\sin(\omega t + \phi)$$

$$a(t) = -\omega^2 x_{max}\cos(\omega t + \phi)$$

Which lead to the realization that: $a = -\omega^2 x$

For each oscillator, we will start with Newton's 2nd Law and try to bring it into the form $\underline{a = -\omega^2 x}$ so that we can deduce what ω is in terms of the physical characteristics of the oscillator.

SHM: The Spring



Find ω so that we can derive an equation for the period, T.

$$\sum F = ma$$

$$-kx = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

diff. equ.

$$a = -\omega^2 x$$

$$ma = -kx$$

$$a = -\frac{k}{m}x$$

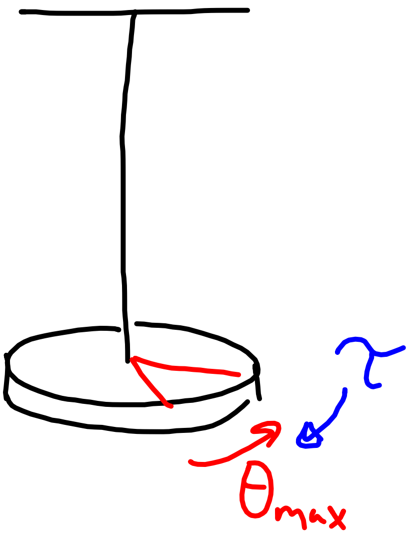
$$\therefore \omega = \sqrt{\frac{k}{m}}$$

$$f = \frac{\omega}{2\pi} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$T = \frac{1}{f} \qquad \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

Torsion Pendulum

The rotational version of a spring



equil pt

restoring τ

$\tau \propto -\theta$

$\tau = -\kappa\theta$

$\left(\frac{mN}{rad}\right)$

Rotating Simple Harmonic Oscillators

have an equilibrium point, and the restoring TORQUE must be proportional to the ANGULAR displacement.

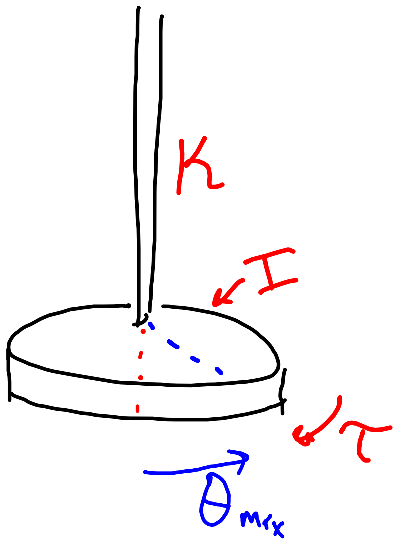
$$\tau = -\kappa\theta$$

$$\theta(t) = \theta_{\max}\cos(\omega t + \varphi)$$

$a = -\omega^2 x$

For each oscillator, we will start with Newton's 2nd Law FOR ROTATION and try to bring it into the form of $\alpha = -\omega^2\theta$ so that we can deduce what ω is in terms of the physical characteristics of the oscillator.

SHM: The Torsion Pendulum



Find ω so that we can derive an equation for the period, T.

$$\sum \tau = I\alpha$$

$$-K\theta = I\alpha$$

$$-K\theta = I\frac{d^2\theta}{dt^2}$$

diff. equ.

$$\alpha = -\omega^2\theta$$

$$I\alpha = -K\theta$$

$$\alpha = -\frac{K}{I}\theta$$

$$\therefore \omega = \sqrt{\frac{K}{I}}$$

$$f = \frac{\omega}{2\pi}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{I}}$$

$$T = \frac{1}{f}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{K}}}$$

A Quick Aside:

Make sure your calculator is in radians mode.

Try sin(1) = 0.841

Try sin(0.5) = 0.479

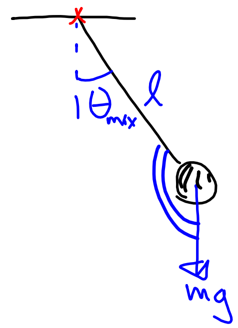
Try sin(0.25) = 0.247

Try sin(0.1) = 0.00998

See a pattern?

For small θ in radians, $\sin\theta \approx \theta$

SHM: The Simple Pendulum



Find ω so that we can derive an equation for the period, T.

$$\sum \tau = I\alpha$$

$$\tau = rF\sin\theta$$
$$= -\ell mg\sin\theta$$

$$-\ell mg\sin\theta = I\alpha$$

small $\theta \rightarrow \sin\theta \approx \theta$

$$-\ell mg\,\theta = I\alpha$$

$$\alpha = -\omega^2\theta$$

$$-\ell mg\,\theta = I\frac{d^2\theta}{dt^2}$$

diff. eqn.

$$I\alpha = -\ell mg\,\theta$$

$$\alpha = -\left(\frac{\ell mg}{I}\right)\theta$$

$$\alpha = -\left(\frac{\ell mg}{m\ell^2}\right)\theta$$

$$\alpha = -\left(\frac{g}{\ell}\right)\theta$$

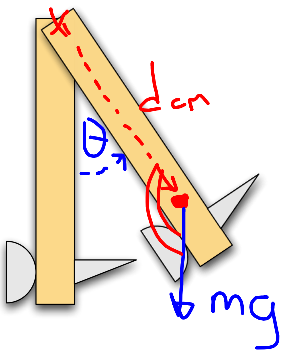
ω^2

$$\therefore \omega = \sqrt{\frac{g}{\ell}}$$

$$f = \frac{\omega}{2\pi} \qquad f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

$$T = \frac{1}{f} \qquad \boxed{T = 2\pi\sqrt{\frac{\ell}{g}}}$$

SHM: The Physical Pendulum



Find ω so that we can derive an equation for the period, T.

$$\sum \tau = I\alpha$$

$$-mgd_{cm}\sin\theta = I\alpha$$

d_{cm} = distance pivot to CM

small $\theta \rightarrow \sin\theta \approx \theta$

$\tau = -d_{cm}\,mg\sin\theta$

$$-mgd_{cm}\theta = I\alpha$$

$\alpha = -\omega^2\theta$

$$-mgd_{cm}\theta = I\frac{d^2\theta}{dt^2}$$

dif eqn.

$$I\alpha = -mgd_{cm}\theta$$

$$\alpha = -\underbrace{\left(\frac{mgd_{cm}}{I}\right)}_{\omega^2}\theta$$

$$\therefore \omega = \sqrt{\frac{mgd_{cm}}{I}}$$

$$f = \frac{\omega}{2\pi}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{mgd_{cm}}{I}}$$

$$T = \frac{1}{f}$$

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgd_{cm}}}}$$

Spring

$$T = 2\pi\sqrt{\frac{m}{k}}$$

k is the spring constant in N/m

m is the mass in kg

$F = -kx$

Torsion Pendulum

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

κ is the torsion constant in mN/rad

I is the rotational inertia in kg m^2

$\tau = -\kappa\theta$

Simple Pendulum *

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Assumes a small mass, far from pivot

l is the length of the pendulum

Physical Pendulum *

$$T = 2\pi\sqrt{\frac{I}{mgd_{cm}}}$$

I is the rotational inertia in kg m^2

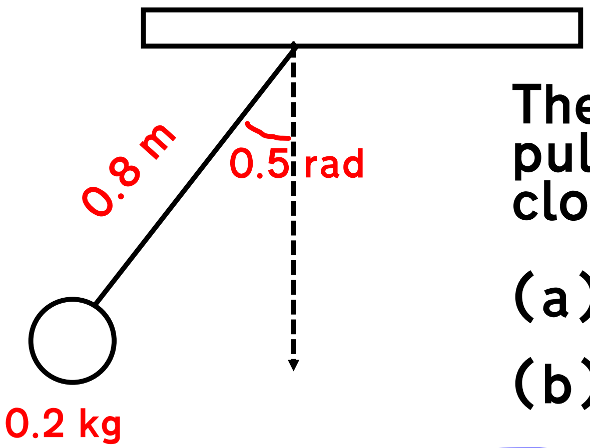
d_{cm} is the distance from the pivot to the CM

m is the mass in kg

Then use $\omega = 2\pi f$ and $T = \dfrac{1}{f}$

* approximately simple harmonic for small displacement angles (30 degrees or so)

EX1 Simple Pendulum



The simple pendulum is pulled back 0.5 rad clockwise and released.

(a) Find T, the period

(b) Write θ(t)

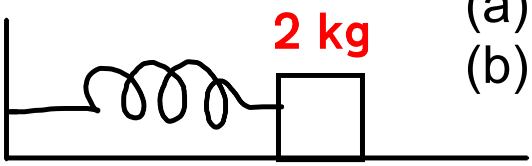
a) $T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{0.8}{10}} = 1.78$ sec.

b) $\theta(t) = \theta_{max}\cos(\omega t + \phi)$

$= (0.5 \text{ rad})\cos(3.54t)$ * not quite right

$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{1.78\text{s}} = 3.54 \frac{\text{rad}}{\text{s}}$

EX2: Spring



(a) Find T, the period
(b) Find k, the spring constant

$x(t) = (0.3\text{ m})\cos(6\pi t)$

a) $\omega = 6\pi \frac{\text{rad}}{\text{s}}$ $\quad f = \frac{\omega}{2\pi}$ $\frac{6\pi}{2\pi} = 3\text{ Hz}$

$T = \frac{1}{f} = \boxed{0.33\text{ sec}}$

b) $T = 2\pi\sqrt{\frac{m}{k}}$

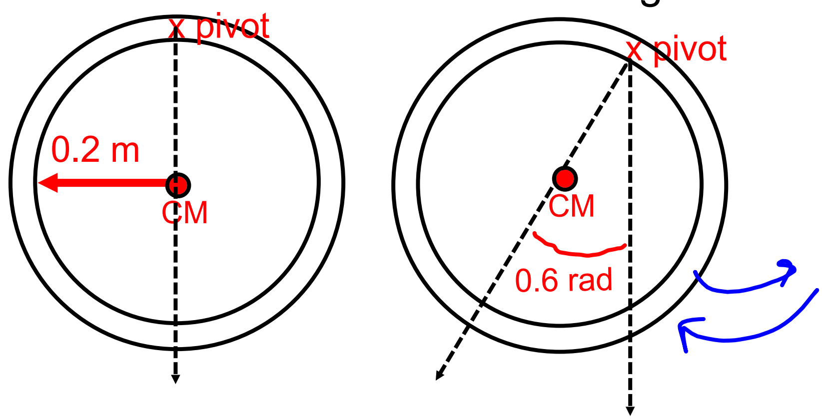
$T^2 = \frac{4\pi^2 m}{k}$

$k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2(2\text{ kg})}{(0.33\text{ s})^2} = \boxed{711\ \frac{\text{N}}{\text{m}}}$

EX3 Physical Pendulum

Ring

Ring pulled back to an angle



Find the frequency of the ring after it is released.

$$T = 2\pi\sqrt{\frac{I}{mgd_{cm}}}$$

$d_{cm} = r$

$I =$ ring, bA

$$I_{new} = I_{cm} + md^2$$
$$= mr^2 + mr^2$$
$$= 2mr^2$$

$$T = 2\pi\sqrt{\frac{2mr^2}{mgr}}$$

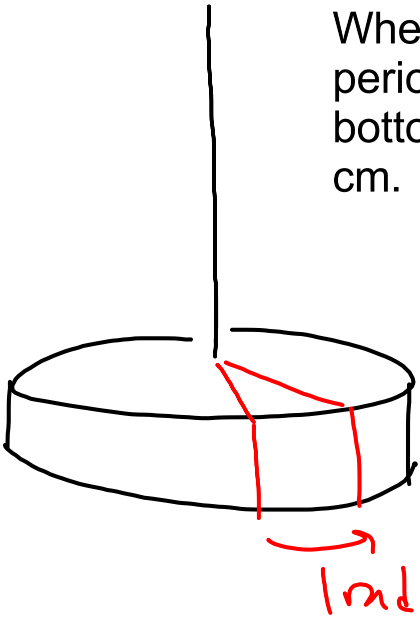
$$T = 2\pi\sqrt{\frac{2r}{g}}$$

$$= 2\pi\sqrt{\frac{2(0.2)}{10}} = 1.26\text{ sec}$$

$$f = \frac{1}{T} = \boxed{0.796\text{ Hz}}$$

EX4 Torsion Pendulum

When pulled back to an angle of 1 radian, the period of the torsion pendulum is 0.6 s. The bottom is a disk with mass 1 kg and radius 10 cm. Find the torsion constant.



$$T = 2\pi\sqrt{\frac{I}{K}}$$

$$T^2 = \frac{4\pi^2 I}{K}$$

$$I = \frac{1}{2}mr^2$$

$$= \frac{1}{2}(1\,kg)(0.1\,m)^2$$

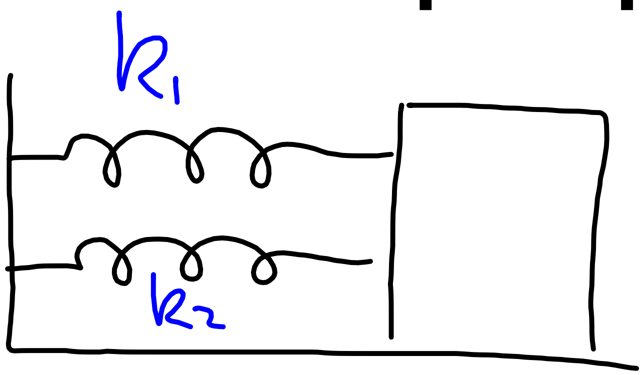
$$= 0.005\,kg\,m^2$$

$$K = \frac{4\pi^2 I}{T^2}$$

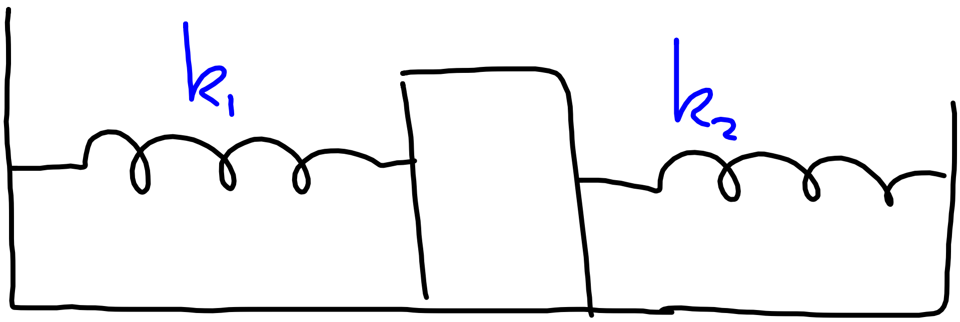
$$= \frac{4\pi^2(0.005)}{(0.6)^2}$$

$$= 0.548\,\frac{mN}{rad}$$

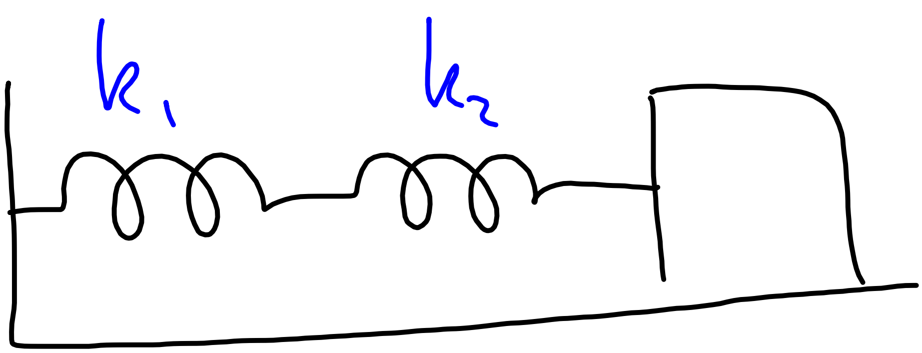
Coupled Springs



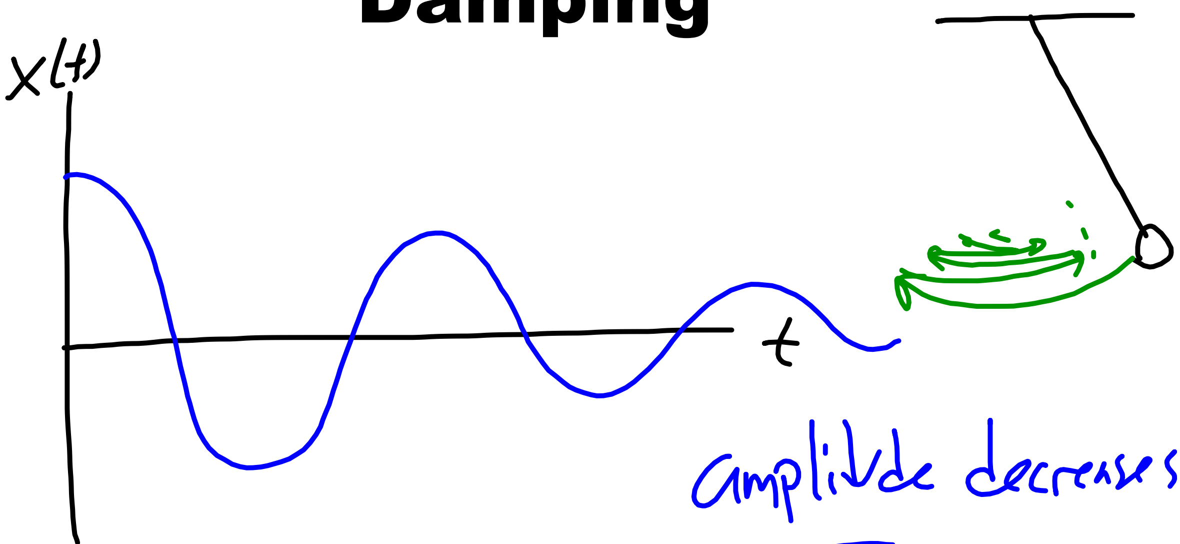
$k_{total} = k_1 + k_2$



$k_{total} \neq k_1 + k_2$



Damping



amplitude decreases

T unchanged

friction + drag.

Natural Frequency = frequency of a system based on its natural properties (like mass, springy-ness, etc.)

Forced Vibration = one vibrating object causes another to vibrate

Resonance = Forced vibration at an object's natural frequency.