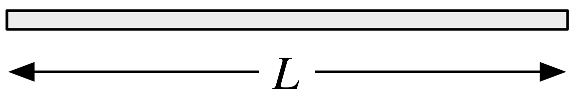
Integrating to get the rotational inertia of a rod around its end.

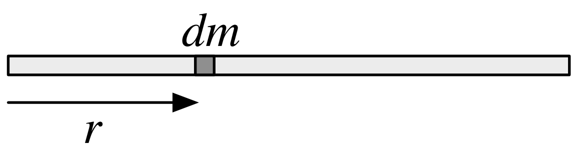
rod with uniform mass per length ρ



$$\rho = \frac{M}{L}$$

$$\rho L = M$$

$$\rho dr = dm$$ (infinitesimally thin piece *dm*)



$$I = \int r^2 \, dm$$ (definition of rotational inertia)

$$I = \int_0^L r^2 (\rho dr)$$ (substituting from above)

$$I = \rho \int_0^L r^2 \, dr$$ (factor out constant)

$$I = \rho \left[\frac{r^3}{3} \right]_0^L$$ (integrating)

$$I = \rho \left[\frac{L^3}{3} - \frac{0^3}{3} \right]$$ (plugging in limits)

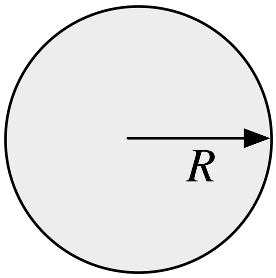
$$I = \frac{\rho L^3}{3}$$

$$I = \frac{(\rho L) L^2}{3}$$ (mass per unit length times total length is total mass)

$$I = \frac{ML^2}{3}$$

Integrating to get the rotational inertia of a disk around its center.

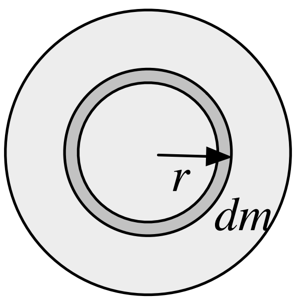
disk with uniform mass per area ρ



$$\rho = \frac{M}{A}$$

$$\rho A = M$$

$$\rho \pi R^2 = M$$



$\rho 2\pi r dr = dm$ (infinitesimally thin ring of mass *dm*)

$I = \int r^2\, dm$ (definition of rotational inertia)

$I = \int_0^R r^2 \left(\rho 2\pi r dr\right)$ (substituting from above)

$I = \rho 2\pi \int_0^R r^3\, dr$ (factor out constants)

$I = \rho 2\pi \left[\frac{r^4}{4}\right]_0^R$ (integrating)

$I = \rho 2\pi \left[\frac{R^4}{4} - \frac{0^4}{4}\right]$ (plugging in limits)

$$I = \frac{\rho 2\pi R^4}{4}$$

$I = \frac{2\left(\rho \pi R^2\right) R^2}{4}$ (mass per unit area times total area is total mass)

$$I = \frac{MR^2}{2}$$